Singular Perturbation Method for Robust Control of Nonlinear Systems

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Abstract — **In this paper, we consider robust control of nonlinear systems, via inclusion nonlinear systems solution and H_∞ controller using singular perturbation method. First, using technique for solving inclusion nonlinear systems, we transform the nonlinear system to ordinary nonlinear system. Then using normal form equations, we eliminate the nonlinear part of the system matrix of equations of system and transform it to a linear diagonal form. Separating new equations to slow and fast subsystems, due to the singular perturbation method and with the assumption of norm-boundedness of the fast dynamics, we can treat them as disturbance and design H_∞ controller for a system with a lower order than the original one that stabilizes the overall closed loop system. The proposed method is applied to a nominal system.**

***Keywords**: Inclusion nonlinear Systems, Robust Control, H_∞ Controller, Singular Perturbation method.*

I. Introduction

In control theory the solution of state feedback nonlinear H_∞ control problem are usually extremely difficult to obtain. if the H_∞ control problem for the linearized system is solvable then locally, one obtains a solution to the nonlinear H_∞ control problem [5]. Also using singular perturbation theory for systems with two (or more) time scales, one can circumvent most of the controller design difficulties for complex multi dimensional systems.

In this paper we consider the robust control of nonlinear systems that is affine in input. First, using technique for solving inclusion nonlinear systems, we transform the nonlinear system to ordinary nonlinear system. Then using normal form equations, we eliminate the nonlinear part of the system matrix of equations of system and transform it to a linear diagonal form. Separating new equations to slow and fast subsystems, due to the singular perturbation method and with the assumption of norm-boundedness of the fast dynamics, we can treat them as disturbance and design H_∞ controller for a system with a lower order than the original one that stabilizes the overall closed loop system. The idea that one can consider the fast dynamics of a singularly perturbed system as disturbances first is discussed in [7]. In fact this is true for many mechanical systems that act as low pass filters. With this idea, we can use the H_∞ method to design a robust controller for the slow subsystem as the nominal plant. In act we consider one part of a system as uncertainty and then design the controller for the remaining certain part of system with an order less than the original one. It is important that choosing a part of system as uncertainty is not optional since the small gain theorem must be hold. The small gain theorem says that a system consists of two subsystems with g1 and g2 gains, is stable if g1.g2<1. Now if we suppose that one of these subsystems is the system related to slow dynamics with he gain g1 , and the other is related to the fast dynamics with the gain g2 , and also consider the subsystem with high frequency dynamics as uncertainty then the whole system is stable when g1.g2<1.With this method we consider the main system via a technique as a system with a dimension less than its actual dimension.

The main contribution of this paper is applying the technique of transform inclusion nonlinear system to ordinary nonlinear system and to use the normal form equations for eliminating the nonlinearities from the system matrix up to the desired degree; using singular perturbation method to separating the dynamic modes of obtained Jordan form equations of system; using the idea stated in [7] for considering the fast modes of system as uncertainty and then designing an H_∞ controller for the slow part of system as the nominal system that stabilizes the whole closed loop system. In section 2, using technique for solving inclusion nonlinear system we transform the nonlinear system to ordinary nonlinear system; use normal form equations to eliminate the nonlinearities of system matrix is stated proposed In section 3. the H_∞ controller is designed for the nominal system in section 4 , and in section 5 the designed H_∞ controller is applied to nominal system as simulation results.

II. INCLUSION SYSTEMS

Consider an inclusion nonlinear system as below:

$$x \in f(x,u) \quad (1)$$

Where $f(x,u)$ is a set function; using technique that proposed in [1]; the inclusion nonlinear systems transformed to ordinary nonlinear systems as:

$$x = f(x) + g(x)u \quad (2)$$

If

$$\sup_u f(x,u) = K(x), \quad \inf_u f(x,u) = L(x) \quad (3)$$

With suppose that

$$x \in [L(x), K(x)] \quad \& \quad \lambda \in (0,1) \quad (4)$$

Then

$$x = \lambda K(x) + (1-\lambda)L(x), \quad x = L(x) + [K(x) - L(x)]\lambda \quad (5)$$

If

$$L(x) = f(x), \quad K(x) - L(x) = g(x) \quad (6)$$

We have:

$$x = f(x) + g(x)\lambda \quad (7)$$

By replace λ with u we have:

$$x = f(x) + g(x)u \quad (8)$$

III. NONLINEAR SYSTEMS

Now, consider nonlinear system as below

$$x = f(x) + g(x)u \quad (9)$$

That we get of a nonlinear system as (8).Using the Taylor expansion of $f(x)$ and $g(x)$ about the equilibrium point (without loss of generality at the origin) and noting that $f(0) = 0$, we have:

$$x = \frac{\partial f}{\partial x}(0)x + \tilde{f}(x) + (g^0 + \tilde{g}(x))u \quad (10)$$

Where

$$\tilde{f}(x) = f(x) - \frac{\partial f}{\partial x}(0)x \quad (11)$$

$$\tilde{g}(x) = g(x) - g^0 \;\&\; g^0 = g(0)$$

Now we use a similarity transformation to transform $\frac{\partial f}{\partial x}(0)$ into Jordan canonical form. With the transformation, $x = Tw$ equation (10) will be:

$$w = T^{-1}\frac{\partial f}{\partial x}(0)Tw + T^{-1}\tilde{f}(Tw) + T^{-1}(g^0 + \tilde{g}(Tw))u \quad (12)$$

Equation (14) Can be written as:

$$w = Jw + F(w) + (G + \tilde{G}(w))u \quad (13)$$

We expand $F(w)$ and $\tilde{G}(w)$ by Taylor series, so that

Equation (13) becomes

$$w = Jw + F_2(w) + F_3(w) + \ldots + F_{r-1}(w) + O(|w|^r) + (G + G_1(w) + G_2(w) + \ldots)u \quad (14)$$

Which $F_i(w)$ and $G_i(w)$ shows the order i term in *w*. We now perform a series of coordinate transformations to eliminate the nonlinearities [6]. The first is

$$y = w - h_2(w)$$

$$v = \alpha_2(w) + (1 + \beta_1(w))u \quad (15)$$

where $h_2(w)$ and $\alpha_2(w)$ are second order functions in *w* and $\beta_1(w)$ is a first order function in *w*. Substituting equation (15) into equation (14) and using the assumption in [6] we will have

$$y = Jy + Gv + [Jh_2(w) - Dh_2(w)Jw + F_2(w) - G\alpha_2(w)] + F_3(w) + \ldots + F_{r-1}(w) + O(|w|^r)$$

$$+ G\beta_1(w)u - Dh_2(w)Gu + G_1(w)u \quad (16)$$

We can choose $h_2(w)$ and $\alpha_2(w)$ and $\beta_1(w)$ as below, so as simplify the second order terms in equation (16)

$$Dh_2(w)Jw - Jh_2(w) + G\alpha_2(w) = F_2(w)$$

$$G\beta_1(w)u + Dh_2(w)Gu = G_1(w)u \qquad (17)$$

We transform equation (16) using

$$y = w - h_3(w)$$

$$v = \alpha_r(w) + (I + \beta_{r-1}(w)) \qquad (18)$$

Where $h_3(w)$ is third order in *w,* and with the same procedure, equation (16) will be transformed to

$$y = Jy + Gv \qquad (19)$$

With an error of order $r+1$. Where *J* is a diagonal matrix of system eigenvalues.

IV. H_∞ CONTROLLER

Consider the system of equation (19). Suppose that we can recognize the slow and fast dynamics of system and decompose it into two subsystems as

$$\begin{aligned} X_1 &= \Lambda_{11}X_1 + B_1u \\ X_2 &= \Lambda_{22}X_2 + B_2u \end{aligned} \qquad (20)$$

Where X_1 is the slow dynamics of system and X_2 is the fast dynamics of system, we must turn it in the main coordination. Now as mentioned earlier we consider the subsystem with high frequency dynamics as uncertainty Δ (Fig. 1).
Suppose Δ, that shows the uncertain dynamics, is asymptotically stable and norm bounded, i.e.

$$\|\Delta\|_\infty \le \gamma_1 \;\; \Delta(s) = (sI - \Lambda_{22})^{-1}B_2 \qquad (21)$$

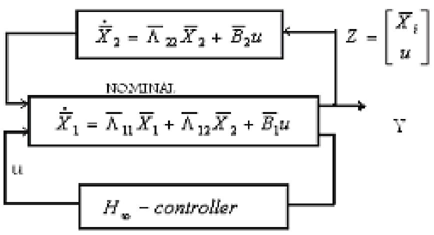

Figure 1. Block diagram of nominal system with uncertain dynamics

P_Δ is the generalized system and consists of Δ and the slow sub-system (nominal). The H_∞ controller design problem for the system of Fig. 1, will be led to a H_∞ controller design for the nominal system that stabilizes it and (or) also perform another object such as reference tracking. To design this controller the only information about the fast subsystem needed is the H_∞ norm of it.
The nominal system can be shown as

$$P \approx \begin{bmatrix} \Lambda_{11} & 0 & B_1 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \qquad (22)$$

Where the controlled output, *Z* is

$$Z = C_1X_1 + D_{12}u \qquad (23)$$

and the output is

$$Y = C_2X_1 + D_{21}X_2 \qquad (24)$$

Since X_2 that shows the fast modes of system and considered as the uncertain dynamics must be entered in the nominal system and can effect on it, we should apply a transformation $[X_1, X_2]^T = M[\bar{X}_1, \bar{X}_2]^T = M\bar{X}$ to the system, then

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} \qquad (25)$$

And

$$\bar{X} = \overline{\Lambda X} + \bar{B}u \qquad (26)$$

Where

$$\bar{\Lambda} = M^{-1}\begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} M = \begin{bmatrix} \bar{\Lambda}_{11} & \bar{\Lambda}_{12} \\ 0 & \bar{\Lambda}_{22} \end{bmatrix}$$

$$\bar{B} = M^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \qquad (27)$$

then the new equation for the fast sub-system will be

$$\bar{X}_2 = \bar{\Lambda}_{22}\bar{X}_2 + \bar{B}_2u \qquad (28)$$

The infinity norm of uncertainty block (fast sub-system) will

$$\overline{\Delta}(s) = M_{22}^{-1}(sI - \overline{\Lambda}_{22})^{-1}\overline{B}_2 = M_{22}^{-1}(sI - M_{22}\Lambda_{22}M_{22}^{-1})^{-1}M_{22}B_2 = \Delta(s) \qquad (29)$$

Thus $\overline{\gamma}_1 = \gamma_1$.
Also new equation for the nominal system will be

$$\overline{P} \approx \begin{bmatrix} \overline{\Lambda}_{11} & \overline{\Lambda}_{12} & \overline{B}_1 \\ \overline{C}_1 & D_{11} & D_{12} \\ \overline{C}_2 & \overline{D}_{21} & 0 \end{bmatrix} \qquad (30)$$

$$Z = \overline{C}_1\overline{X}_1 + D_{11}\overline{X}_2 + D_{12}u$$
$$Y = \overline{C}_2\overline{X}_1 + \overline{D}_{21}\overline{X}_2 \qquad (31)$$

Where

$\overline{C}_2 = C_2M_{22}^{-1}$, $\overline{C}_1 = C_1M_{11}^{-1}$
$\overline{D}_{21} = D_{21} - C_2M_{11}^{-1}M_{12}M_{22}^{-1}$,
$D_{11} = -C_1M_{11}^{-1}M_{12}M_{22}^{-1}$

The∞ H_∞ controller can be design via state feedback or output feedback. If we suppose that all states of the system are accessible, then we use state feedback. We must determine $\overline{\gamma}_2 = \min \gamma$, such that the eigenvalues of Hamiltonian matrix.

$$H = \begin{bmatrix} \overline{\Lambda}_{11} & \gamma^{-2}\overline{\Lambda}_{12}\overline{\Lambda}_{12}^T - \overline{B}_1\overline{B}_1^T \\ -\overline{C}_1^T\overline{C}_1 & -\overline{\Lambda}_{11}^T \end{bmatrix} \qquad (32)$$

Don't place on the imaginary axis. Then we must find M, such that $\overline{\gamma}_1.\overline{\gamma}_2 < 1$.Under assumption of stabilizability-detectability of reference [7], for a given $\gamma > 0$, there is an internal stabilizing controller such that $\left\| T_{z\overline{X}_2} \right\|_\infty \le \gamma$, if and only if X_∞ is a p.s.d. solution of algebraic Riccati equation

$$\overline{\Lambda}_{11}^T X_\infty + X_\infty\overline{\Lambda}_{11} + X_\infty(\gamma^{-2}\overline{\Lambda}_{12}\overline{\Lambda}_{12}^T - \overline{B}_1\overline{B}_1^T)X_\infty + \overline{C}_1^T\overline{C}_1 = 0 \quad (33)$$

Then the related controller will be in the form

$$u(t) = -\overline{B}_1^T X_\infty \overline{X}_1(t) \qquad (34)$$

V. SIMULATION RESULTS

Consider the equations as follows:

$$x = \begin{bmatrix} x_{12} \\ -20.4\sin x_{11} - 23.53x_{21} \\ 10x_{22} \\ -20.4\sin x_{11} + 1.73x_{12} - 201.7x_{21} - 1.75x_{22} + 87.5\sin(3u) \end{bmatrix}$$

Where

$$x = [x_{11}, x_{12}, x_{21}, x_{22}]^T$$

Using (3)-(7) this nonlinear system transformed to below system that ordinary nonlinear system:

$$x = f(x) + g(x)u$$

Where

$$f(x) = \begin{bmatrix} x_{12} \\ -20.4\sin x_{11} - 23.53x_{21} \\ 10x_{22} \\ -20.4\sin x_{11} + 1.73x_{12} - 201.7x_{21} - 1.75x_{22} - 87.5 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 175 \end{bmatrix}$$

This system has an equilibrium point at the origin, also $f(0) = 0$. The object is to design a controller that the position of x_{11} reaches from zero state to zero. Using the Taylor expansion about the equilibrium point we can write

$$x = \frac{\partial f}{\partial x}(0)x + \tilde{f}(x) + (g^0 + \tilde{g}(x))u$$

Using the transformation $x = Tw$ we transform

$$x = Ax + Bu \Rightarrow w = T^{-1}ATw + T^{-1}Bu = Jw + T^{-1}Bu$$

where we have:

$$J = \begin{bmatrix} -.7722 & 44.9227 & 0 & 0 \\ -44.9227 & -.7722 & 0 & 0 \\ 0 & 0 & -.1028 & 4.2416 \\ 0 & 0 & -4.2416 & -.1028 \end{bmatrix}$$

$$T^{-1}B = \begin{bmatrix} 180.2418 \\ -3.9377 \\ -21.152 \\ .4524 \end{bmatrix}$$

Noting that the eigenvalues of J belong to two clusters (fast and slow), we can separate the whole system to two subsystems as follows,

The slow sub-system:

$$w_s = J_s w_s + F_s(w) + G_s u,$$

$J_s = diag\{-.1028 \pm 4.2416i\},$
$F_s(w) = [F_{s1}(w), F_{s2}(w)]^T,$
$G_s = [G_{s1}, G_{s2}].$

The fast sub-system:

$$w_f = J_f w_f + F_f(w) + G_f u,$$
$$J_f = diag\{-.7722 \pm 44.9227i\},$$
$$F_f(w) = [F_{f1}(w), F_{f2}(w)]^T,$$
$$G_f = [G_{f1}, G_{f2}].$$

The fast subsystem is asymptotically stable (noting its eigenvalues). Thus it can be considered as uncertainty block in Fig. 1. using transformation $X = M\overline{X}$, with $M_{11} = M_{22} = I, M_{12} = .133, M_{22} = 0$ and theorem 1, we apply H_∞ controller of equation (29) to the system, then the regulation of slow modes of system and the behavior of fast dynamics that are modeled as uncertainty the controller output is shown in Fig, 2, Fig. 3 shows output system tracking with sinusoid and impulse input.

VI. CONCLUSION

In this paper the robust control of nonlinear systems via singular perturbation theory is considered. First using the technique of solving inclusion nonlinear systems that proposed we transformed the nonlinear system to ordinary nonlinear system. Then by normal form theory the nonlinearity of system matrix is eliminated up to the desired degree. With the assumption about norm-boundedness of the fast dynamics and considering them as uncertainty, we have designed H_∞ controller for a reduced order system (slow subsystem) but the whole closed loop system will be stable.

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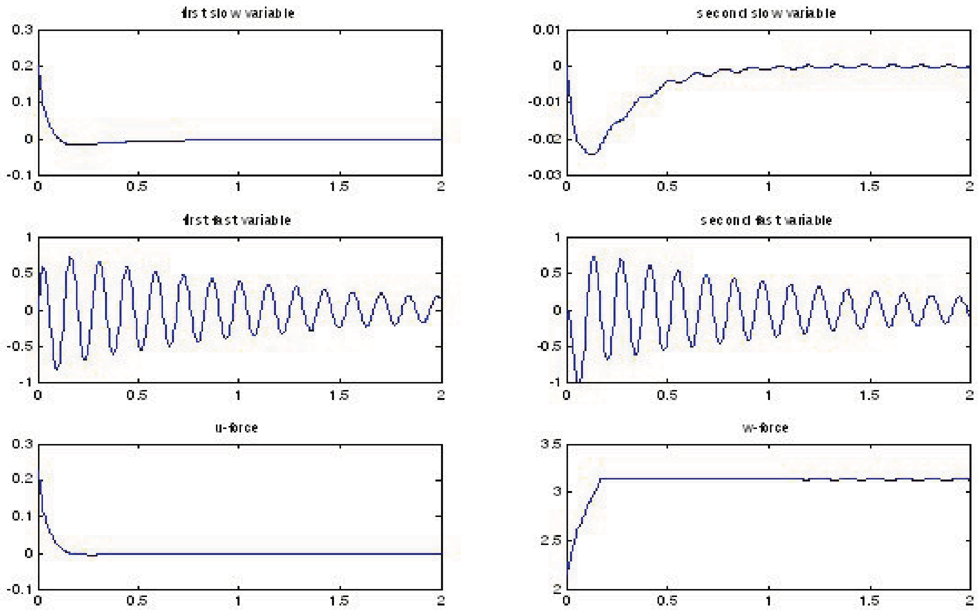

Figure 2. Regulation of slow & fast dynamics and controller u & λ

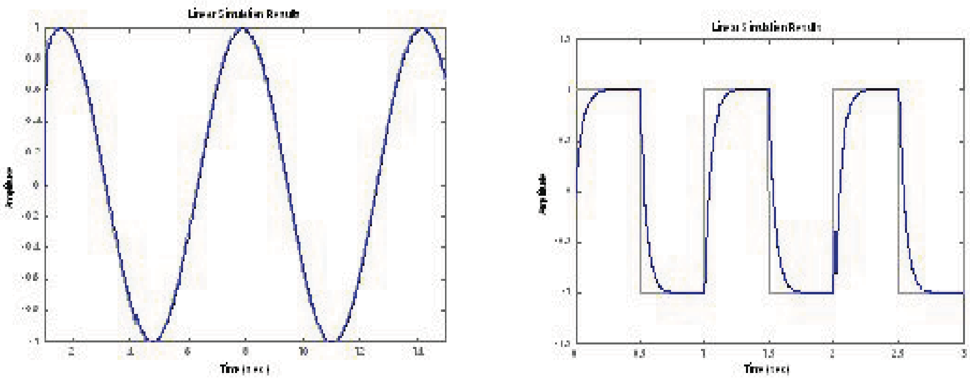

Figure 3. Tracking of slow dynamic system with sinusoid & impulse inputs .